REGULARITY AND RIGIDITY OF ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. In this paper, we study some intrinsic characterization of conformally compact manifolds. We show that, if a complete Riemannian manifold admits an essential set and its curvature tends to -1 at infinity in certain rate, then it is conformally compactifiable and the compactified metrics can enjoy some regularity at infinity. As consequences we prove some rigidity theorems for complete manifolds whose curvature tends to the hyperbolic one in a rate greater than 2.

1. introduction

In recent years there are growing interests in the study of conformally compact Riemannian manifolds from mathematics and physics. Conformally compact Einstein manifolds, for instance, are the basic objects that are used to establish the mathematical foundation for the so-called AdS/CFT correspondence proposed and studied in some promising theory of quantum gravity.

Suppose that X^{n+1} is a smooth manifold with boundary $\partial X = M^n$. A defining function x of the boundary M^n in X^{n+1} is a smooth function on X^{n+1} such that

(1) $x > 0$ in X^{n+1};
(2) $x = 0$ on M^n;
(3) $dx \neq 0$ on M^n.

A complete Riemannian metric g on X^{n+1} is said to be conformally compact of regularity $C^{k,\alpha}$ if $x^2 g$ extends to be a $C^{k,\alpha}$ compact Riemannian metric on $\bar{X}^{n+1}$ for a defining function x of the boundary M^n in X^{n+1}.

A basic and interesting question is that, what are the sufficient conditions for a complete Riemannian manifold $(X^{n+1},\ g)$ to be conformally compact of reasonable regularity? It is rather easy to see that the Riemann curvature

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needs to tend to a negative constant at the infinity. On the other hand, due to the complexity of the end structure of a hyperbolic manifold, it is clear that a simple volume growth condition would not be enough to yield anything like what is true about asymptotically locally Euclidean manifolds.

With those understandings in mind, in this note, as in [17], we will consider a complete, noncompact Riemannian manifold $(X,\ g)$ whose curvature is asymptotically hyperbolic of order a as follows:

$$||Rm - \mathbf{K}|| \leq Ce^{-a\rho} \tag{1}$$

where Rm denotes the Riemann curvature tensor of the metric g and $\mathbf{K}$ the constant curvature tensor of -1, i.e., $\mathbf{K}_{ijkl} = -(g_{ik}g_{jl} - g_{ij}g_{kl})$, ρ is the distance function to a fixed point in X with respect to g, and C is a positive constant independent of ρ. To control the wild behavior of the ends we consider a notion of essential sets which was introduced in [4] and [2].

Definition 1.1. *A compact subset* $\mathbf{E}$ *of* $(X,\ g)$ *is called an essential set if*

(1) $\mathbf{E}$ *is a compact domain of* X *with smooth and convex boundary* $\mathbf{B} := \partial\mathbf{E}$, *i.e. its second fundamental forms with respect to the outward unit normal vector field is positive definite. Any geodesic half line emitting from* $\mathbf{B}$ *orthogonally to the outside of* E *can be extended to infinity;*

(2) *the distance function* ρ *to the essential set is smooth;*

(3) *the region in* X *which is outside the essential set* $\mathbf{E}$ *is diffeomorphic to* $[0, \infty) \times \mathbf{B}$.

It is not easy to determine whether or not there is an essential set even in a complete hyperbolic manifold. But it clearly is a necessary condition for a complete Riemannian manifold to be conformally compactifiable in the above sense. Indeed in [2] and [3], for a complete Riemannian manifold which possesses an essential set and whose curvature is asymptotically hyperbolic and covariant derivatives of Riemann curvature decay, the authors were able to obtain conformal compactifications with some regularity results. Those decay assumptions on the covariant derivatives of curvature were derived when in addition the manifold is Einstein (cf. [3]).

Now suppose that $(X^{n+1},\ g)$ is a complete Riemannian manifold that possesses an essential set $\mathbf{E}$. Then we know that outside $\mathbf{E}$ the metric g can be written as

$$g = d\rho^2 + g_{ij}(\rho, \theta)d\theta^i d\theta^j,$$

where ρ is the distance function to $\mathbf{E}$ and θ is a local coordinate on $\mathbf{B} = \partial\mathbf{E}$. We have a convention for indices in this note that all Latin letters runs from 1 to n while all Greek letters runs from 0 to n. With this identification there is

a natural differential structure on the closure $\bar{X}$ which simply is the product structure as follows. Let

$$\mathbf{E}_1 = \{x \in M : \rho(x) \leq 1\},$$

then $\mathbf{E} \subset \mathbf{E}_1$ and

$$\bar{X} = \mathbf{E}_1 \coprod [0, \delta_0] \times \mathbb{B},$$

where $\delta_0 = \log \frac{e+1}{e-1}$, and $\coprod$ denotes the connected sum by identifying $(1, \theta) \in \partial \mathbf{E}_1$ with $(\delta_0, \theta) \in \{\delta_0\} \times \mathbb{B}$. Hence we may use the coordinate (τ, θ) to replace (ρ, θ) outside $\mathbf{E}_1$ such that

$$\tau = \log \frac{e^\rho + 1}{e^\rho - 1} \text{ and } \sinh^{-1} \rho = \sinh \tau.$$

Therefore, in the new coordinate, we consider

$$\begin{aligned}
\bar{g} &= \sinh^{-2} \rho \cdot g \\
&= \sinh^{-2} \rho d\rho^2 + \sinh^{-2} \rho g_{ij}(\rho, \theta) d\theta^i d\theta^j \\
&= d\tau^2 + \bar{g}_{ij}(\tau, \theta) d\theta^i d\theta^j,
\end{aligned}$$

Now $(X \setminus \mathbf{E}_1, \ \bar{g})$ is a conformal compactification of $(X \setminus \mathbf{E}_1, \ g)$ with the defining function $\sinh \tau$ of the boundary $\mathbf{B}$ in $\bar{X}$. Thus we will focus on the regularity of $\bar{g}$ at $\tau = 0$. Note that perhaps we will have to consider it in other coordinate systems nearby the infinity in order to get better regularity.

As pointed out in [3] the regularity of their conformal compactifications were obtained via ODE analysis. The draw back for that is the demand of the assumptions on the decay of covariant derivatives of Riemann curvature such as:

$$||\nabla^k Rm|| \leq C e^{-b\rho}, \tag{2}$$

where C is a positive constant independent of ρ. For instance, in [3], it assumes (2) for $k = 1$ and $b > 0$ to have $C^{0,b}$ regularity; and it assumes (2) for $k = 2$ and $b > 1$ to have $C^{1,b-1}$ regularity. Note that, if the curvature condition (1) holds with $a > 0$ and the curvature condition (2) holds with $b > 0$, then $a \geq b$, as noticed in [3]. Our first goal in this note is to make use of harmonic coordinates near the infinity. We are able to obtain the $C^{2,\alpha}$ regularity under the curvature condition (2) with $k = 1$ and $b > 2$.

Theorem 1.2. *Suppose that $(X^{n+1}, \ g)$ is a complete Riemannain manifold with an essential set $\mathbf{E}$ and that it satisfies the curvature condition (1) with $a > 0$ and the curvature condition (2) with $k = 1$ and $b > 2$. Then there is differentiable structure on $\bar{X}$, which is smooth in the interior of X and $C^{3,\alpha}$ up to the boundary for some $\alpha \in (0, 1)$. And in this differentiable structure, $\bar{g}$ is smooth in the interior of X and is $C^{2,\alpha}$ smooth up to the boundary for some $\alpha \in (0, 1)$.*

We remark here that in fact we can get $C^{1,\alpha}$ regularity when $b > 2 - \frac{1}{n+1}$ (please see Theorem 3.5 in Section 3). Our second goal in this note is to make use of the Ricci flow on complete manifolds to obtain some regularity without assuming decay conditions on the covariant derivatives of curvature. Let $(X^{n+1},\ g)$ be a complete Riemannian manifold satisfying the curvature condition (1) with $a > 0$. We consider the normalized Ricci flow as follows:

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\beta} = -2n g_{\alpha\beta} - 2R_{\alpha\beta}, \\ g_{\alpha\beta}(x,0) = g_{\alpha\beta}(x), \quad \text{on} \quad X. \end{cases} \tag{3}$$

Since our initial metric satisfies the curvature condition (1), by the works of Shi (see Theorem 1.1 in [16]), the evolution equation (3) has a smooth solution $g(\cdot,t)$ for a short time. Let

$$E_{\alpha\beta\gamma\delta}(g) = R_{\alpha\beta\gamma\delta}(g) + (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}).$$

Using the evolution equations and the maximum principles we show that

Lemma 1.3. *Suppose that $(X^{n+1},\ g)$ is a complete Riemannian manifold and that it satisfies the curvature condition (1) with $a > 0$. Let $(M, g(t))$, $t \in [0,T]$, be a complete solution of the normalized Ricci flow (26). Then there exists constants T_0 and C such that*

$$\|E(g(t))\| \leq Ce^{-a\rho_0}$$

and

$$\|\nabla E(g(t))\| \leq \frac{C}{\sqrt{t}} e^{-a\rho_0},$$

where ∇ is with respect to metric $g(t)$, and C is independent of t, $0 < t \leq T_0 \leq T$.

It is then rather easy to obtain an improvement of Theorem A in [3] as follows:

Theorem 1.4. *Suppose $(X^{n+1},\ g)$ is a complete Riemannian manifold with an essential set* **E** *and that it satisfies the curvature condition (1) with $0 < a < 1$. Then $\bar{g}$ is $C^{0,\mu}$ smooth up to the boundary $\tau = 0$ for $\mu = \frac{2}{3}a$.*

We observe that the curvature estimates for the compactified metrics $\bar{g}$ depends only on the curvature condition (1), even though our constructions of harmonic coordinates used the curvature condition (2). Based on compactness theorems in [12] of Riemannian manifolds we find the regularity of harmonic coordinates indirectly via some good approximations provided by the Ricci flow. We then obtain

Theorem 1.5. *Suppose $(X^{n+1},\ g)$ is a complete Riemannian manifold with an essential set* $\mathbf{E}$ *and that it satisfies the curvature condition (1) with $a > 2 - \frac{1}{n+1}$. Then there is differentiable structure Γ on $\bar{X}$, which is smooth in the interior of X and $W^{3,p}$ up to the boundary for some $p > n+1$. And in this differentiable structure Γ, $\bar{g}$ is smooth in the interior of X and it is $W^{2,p}$ up to the boundary for some $p > n+1$, hence, it is $C^{1,\mu}$ smooth up to the boundary for some $\mu \in (0,1)$. Moreover $\bar{g}$ is smooth in the interior of X and it is $W^{2,p}$ up to the boundary for any $p > 1$, hence, it is $C^{1,\mu}$ smooth up to the boundary for any $\mu \in (0,1)$ when $a \geq 2$.*

One of the motivation to derive an intrinsic criterion for a complete Riemannian manifold to be conformally compactifiable is to find a rigidity theorem for a complete Riemannian manifold whose curvature is asymptotically hyperbolic as in the work of Shi and Tian [17]. Although many interesting rigidity theorems were obtained (e.g. [1], [6],[5],[10] [15] [17]) lately, most of them need to assume some regularity of the comformally compactness of manifolds at infinity. As consequences of our regularity theorems we will state and prove Theorem 5.1 and Theorem 5.3 in §5. Theorem 5.1 is a rigidity theorem for Einstein AH manifolds; while Theorem 5.3 requires the curvature condition (2). And in both cases the rigidity in higher dimensions requires the spin condition.

Very recently, in [10], a rigidity theorem for asymptotically hyperbolic manifold with $Ric \geq -ng$ and admitting C^2 conformal compactification was proved. We find that our curvature estimate (1) in Proposition 2.2 and the above Theorem 1.5 together is a perfect substitute for the regularity assumption in the rigidity theorem of [10]. Our main rigidity theorem is as follows:

Theorem 1.6. *Let $(X^{n+1},\ g)$ be a complete manifold with $Ric \geq -ng$. Suppose that it has an essential set* $\mathbf{E}$ *and it satisfies the curvature condition (1) with $a > 2$. Assume also that X^{n+1} is simply connected at the infinity. Then $(X^{n+1},\ g)$ is a standard hyperbolic space for $n \geq 4$. And it is a standard hyperbolic space if in addition we assume that $\int_X \|Rm - \mathbf{K}\| d\mu_g < \infty$ for $n = 3$.*

The rest of the paper is organized as follows. In §2, we will present some basic estimates which can be derived mostly just from the Riccati equations and ODE analysis. In §3, we construct harmonic coordinates at the infinity, therefore prove Theorem 1.2 via the estimates for the curvature of the compactified metrics. In §4, we drop the curvature condition (2) by using Ricci flow. Finally in §5, we prove rigidity results.

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2. Basic estimates

In this section we present the basic estimates. First we introduce some estimates that are consequences of the Riccati equations via ODE analysis as Lemma 2.3 in [17] (see also in [2], [3]). Suppose that $(X^{n+1},\ g)$ is a complete Riemannian manifold with an essential set $\mathbf{E}$. And suppose that (1) holds. Let Σ_ρ be the level surface of the distance function ρ to the essential set $\mathbf{E}$. For simplicity we may take orthonormal frames on each slice Σ_ρ, under which the second fundamental forms are denoted by h_{ij}, then we recall Riccati's equations:

$$\frac{\partial h_{ij}}{\partial \rho} + h_{ik}h_{kj} = R_{0i0j},$$

where the index 0 refers to unit normal direction of Σ_ρ.

Lemma 2.1. *Suppose that $f(\rho) \geq \frac{1}{4}$ is a smooth function for $\rho > 0$ and that*

$$|f(\rho) - 1| \leq Le^{-a\rho},$$

for any $\rho > 0$ and some $a, L > 0$. If y is the solution of the equation

$$\begin{cases} y' + y^2 = f \\ y(0) > 0 \end{cases},$$

then there is a positive constant C which depends only on L and $y(0)$ such that

(1) $|y - 1| \leq Ce^{-2\rho}$, *if $a > 2$;*
(2) $|y - 1| \leq C\rho e^{-2\rho}$, *if $a = 2$;*
(3) $|y - 1| \leq Ce^{-a\rho}$, *if $0 < a < 2$.*

Proof. The proof is very similar to the proof of Lemma 2.3 in [17]. First it is easy to see that

$$0 < y < C, \tag{4}$$

for some constant C, which only depends on L and $y(0)$. Let

$$y = v + 1.$$

Then we get

$$v' + 2v = (f - 1) - v^2$$

and

$$(v^2)' + 2(2 + v) \cdot v^2 = 2v \cdot (f - 1).$$

Hence

$$(v^2)' + 2v^2 < (v^2)' + 2(2+v) \cdot v^2 = 2v \cdot (f-1).$$

Therefore, in the light of (4) and the assumptions on the function f, it follows that

$$|v|^2 \leq Ce^{-2\rho}, \ if \ a > 2,$$
$$|v|^2 \leq C\rho e^{-2\rho}, \ if \ a = 2,$$
$$|v|^2 \leq Ce^{-a\rho}, \ if \ 0 < a < 2.$$

Plugging back those into the same equation we have the improved estimates for v^2. Finally we finish the proof of Lemma 2.1 using the equation

$$v' + 2v = (f-1) - v^2.$$

□

Consequently, if we write

$$h_{ij} = \delta_{ij} + T_{ij}e^{-2\rho},$$

we have

$$|T_{ij}| \leq C, \quad \text{if} \quad a > 2; \tag{5}$$

$$|T_{ij}| \leq C\rho, \quad \text{if} \quad a = 2; \tag{6}$$

and

$$|T_{ij}| \leq Ce^{(2-\alpha)\rho}, \quad \text{if} \quad 0 < a < 2, \tag{7}$$

Note that the constants C in the above depend only on the second fundamental forms of the level surface Σ_0 and the constant in the assumption (1). Similarly we also have the estimates of the second fundamental forms S_{ij} of Σ_ρ under local coordinates $\{\frac{\partial}{\partial\theta^i}\}_{i=1}^n$ on $\mathbf{B}$. Again if we write

$$S_i^j = g^{kj}S_{ik},$$

and

$$S_i^j = \delta_i^j + p_i^j e^{-2\rho},$$

then

$$|p_i^j| \leq C, \quad \text{if} \quad a > 2; \tag{8}$$

$$|p_i^j| \leq C\rho, \quad \text{if} \quad a = 2; \tag{9}$$

and

$$|p_i^j| \leq Ce^{(2-\alpha)\rho}, \quad \text{if} \quad 0 < a < 2. \tag{10}$$

Using those estimates on the second fundamental form of the level set the distance function ρ one can easily derive the following estimates on the curvature of the compactified metric $\bar{g}$ as follows:

Proposition 2.2. *Let $(X^{n+1},\ g)$ be an complete Riemannian manifold with an essentail set $\mathbf{E}$ and satisfying the curvature condition (1) for $a > 0$. If $g = d\rho^2 + g_{ij}d\theta^i d\theta^j$, and $\bar{g} = \sinh^{-2}\rho g := d\tau^2 + \bar{g}_{ij}d\theta^i d\theta^j$, then we have*

(1) $\|Rm(\bar{g})\|_{\bar{g}} \leq \Lambda$, *if* $a > 2$;
(2) $\|Rm(\bar{g})\|_{\bar{g}} \leq \Lambda\rho$, *if* $a = 2$;
(3) $\|Rm(\bar{g})\|_{\bar{g}} \leq \Lambda e^{(2-a)\rho}$, *if* $0 < a < 2$.

If in addition we assume the curvature condition (2) for $k = 1$ and $1 < b < 3$, then we also have

$$\|\bar{\nabla} Rm(\bar{g})\|_{\bar{g}} \leq \Lambda e^{(3-b)\rho}.$$

Here Λ is a constant depending only on C in the assumptions (1) and (2).

Proof. Let $\bar{S}$ be the second fundamental form of Σ_ρ in $(X^{n+1},\ \bar{g})$. Since there exists a constant Λ such that in local coordinates $\{\mathcal{U}, (\rho, \theta^1, ..., \theta^n)\}$,

$$\frac{1}{\Lambda}e^{2\rho}\delta_{ij} \leq g_{ij} \leq \Lambda e^{2\rho}\delta_{ij},$$

we have

$$|\bar{g}_{ij}| \leq C.$$

Then by a direct computation we get

$$\partial_\tau \bar{g}_{ij} = 2(\frac{\cosh\rho}{\sinh^2\rho}g_{ij} - \frac{S_{ij}}{\sinh\rho}) = -2\bar{S}_{ij}.$$

Hence we have

$$\bar{S}_{ij} = \frac{e^{-2\rho}}{\sinh\rho}p_{ij} - e^{-\rho}\bar{g}_{ij}$$

and

$$\bar{S}_i^j = e^{-2\rho}\sinh\rho p_i^j - e^{-\rho}\delta_i^j.$$

Using Gauss identity, we want to express $\bar{R}^l_{ijk}$ in terms of E^l_{ijk} and p_i^j as follows:

$$\bar{R}^l_{ijk} = e^{-2\rho}(\delta^l_j\bar{g}_{ik} - \delta^l_i\bar{g}_{jk}) + E^l_{ijk} - e^{-2\rho}\coth\rho(p_{ik}\delta^l_j + p^l_j g_{ik} - p^l_i g_{jk} - p_{jk}\delta^l_i).$$

On the other hand, we may calculate directly

$$\bar{R}^l_{0jk} = -\sinh\rho\ E^l_{0jk}$$

and

$$\bar{R}^k_{0i0} = \sinh^2\rho\cdot E^k_{0i0} - e^{-2\rho}\cdot\sinh\rho\cdot\cosh\rho\cdot p^k_i + e^{-\rho}\cdot\cosh\rho\cdot\delta^k_i.$$

Those readily imply the estimates for the curvature of the compactified metric $\bar{g}$ with the estimates (8), (9) and (10).

To obtain the estimate on the normal derivatives of the second fundamental form $\bar{S}$, we derive from the Riccati equations that

$$\partial_\rho p_i^j = e^{2\rho} E_{0i0}^j - e^{-2\rho} p_i^k p_k^j,$$

which implies

$$|\partial_\rho p_i^j| \leq \Lambda e^{(2-a)\rho}. \tag{11}$$

To obtain the estimate on the tangential derivatives of the 2nd fundamental form $\bar{S}$, we will take derivatives in the two sides of the Riccati's equation. But first

$$R_{0i0,k}^j = \partial_k R_{0i0}^j - R_{li0}^j \Gamma_{0k}^l - R_{0l0}^j \Gamma_{ik}^l - R_{0il}^j \Gamma_{0k}^l + R_{0i0}^l \Gamma_{lk}^j,$$

which implies

$$|\partial_k R_{0i0}^j| \leq \Lambda e^{-(b-1)\rho},$$

where we have used the fact that the compactified metric $\bar{g}$ is Lipschitz when $k = 1$ and $b > 1$ in (2) and $a > 0$ in (1) (cf. [2] and [3]). Then from Riccati's equations, we have

$$\partial_\rho \partial_k S_i^j + \partial_k S_i^l \cdot S_l^j + \partial_k S_l^j \cdot S_i^l = \partial_k R_{0i0}^j.$$

Considering the ODE system

$$y' + (2 + \Omega)y = O(e^{-(b-1)\rho}),$$

where

$$|\Omega| \leq \begin{cases} \Lambda e^{-2\rho}, a > 2, \\ \Lambda \rho e^{-2\rho},\ a = 2, \\ \Lambda e^{-a\rho},\ 0 < a < 2, \end{cases}$$

and an argument similar to the one in the proof of Lemma 2.1, we then have

$$|\partial_k S_i^j| \leq \Lambda e^{-(b-1)\rho}, \tag{12}$$

where Λ is a positive constant independent of ρ and θ. Hence

$$|\partial_k p_i^j| \leq \Lambda e^{-(b-1)\rho}. \tag{13}$$

Now we are ready to get the estimates on the covariant derivatives of curvature by direct computations.

$$
\begin{aligned}
\bar{R}^l_{ijk,m} = & E^l_{ijk,m} - \frac{\cosh\rho}{\sinh\rho}(E_{0kji}\delta^l_m - E^l_{0jk}g_{mi} + E^l_{0ik}g_{mj} - E^l_{ij0}g_{mk}) \\
& - \frac{e^{-2\rho}\cosh\rho}{\sinh\rho}(\partial_m p_{ik}\delta^l_j - \partial_m p_{jk}\delta^l_i + p^n_j g_{ik}\Gamma^l_{mn} - p^n_i g_{jk}\Gamma^l_{mn} \\
& - p_{nk}\delta^l_j\Gamma^n_{mi} + p_{nm}\delta^l_i\Gamma^n_{mj} + p^l_n g_{jk}\Gamma^n_{mi} - p^l_n g_{ik}\Gamma^n_{mj} \\
& - p_{in}\delta^l_j\Gamma^n_{mk} + p_{jn}\delta^l_i\Gamma^n_{mk})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\bar{R}^l_{ijk,0} = & -\sinh\rho E^l_{ijk,0} - 2\cosh\rho E^l_{ijk} + \frac{2e^{-3\rho}}{\sinh\rho}(\delta^l_j g_{ik} - \delta^l_i g_{jk}) \\
& + (\frac{e^{-4\rho}}{\sinh\rho} - 2e^{-3\rho}\cosh\rho)(p_{ik}\delta^l_j + p^l_j g_{ik} - p^l_i g_{jk} - p_{jk}\delta^l_i) \\
& + 2e^{-2\rho}\cosh\rho(p^l_j g_{ik} - p^l_i g_{jk}) - 2e^{-4\rho}\cosh\rho(p_{mk}p^m_i\delta^l_j - p_{mk}p^m_j\delta^l_i) \\
& + e^{-2\rho}\cosh\rho(\partial_\rho p_{ik}\delta^l_j + \partial_\rho p^l_j g_{ik} - \partial_\rho p^l_i g_{jk} - \partial_\rho p_{jk}\delta^l_i)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\bar{R}^l_{0jk,m} = & -\sinh\rho\cdot E^l_{0jk,m} + \cosh\rho\cdot(E^0_{0jk}\delta^l_m - E^l_{mjk} - E^l_{0j0}g_{mk}) \\
& + (\frac{e^{-3\rho}}{\sinh^2\rho} - \frac{e^{-2\rho}\cosh\rho}{\sinh^2\rho})g_{jk}\delta^l_m - (\frac{e^{-3\rho}}{\sinh^2\rho} + \frac{e^{-2\rho}\cosh\rho}{\sinh^2\rho})g_{mk}\delta^l_j \\
& + (\frac{e^{-4\rho}}{\sinh\rho} + \frac{2e^{-3\rho}\cosh\rho}{\sinh\rho})p_{mk}\delta^l_j \\
& - \frac{e^{-4\rho}}{\sinh\rho}g_{jk}p^l_m + \frac{2e^{-3\rho}\cosh\rho}{\sinh\rho}g_{mk}p^l_j \\
& - e^{-4\rho}\cosh\rho(p_{nk}p^n_m\delta^l_j - g_{jk}p^n_m p^l_n + 2p^l_{jb}p_{mk}).
\end{aligned} \tag{16}
$$

$$
\bar{R}^l_{0jk,0} = \sinh^2\rho\cdot E^l_{0jk,0} + 2\sinh\rho\cdot\cosh\rho\cdot E^l_{0jk} \tag{17}
$$

$$
\begin{aligned}
\bar{R}^l_{0j0,m} = & \sinh^2\rho\cdot E^l_{0j0,m} + \sinh\rho\cdot\cosh\rho(E^l_{mj0} + E^l_{0jm}) \\
& - e^{-2\rho}\sinh\rho\cdot\cosh\rho\cdot\partial_m p^l_j \\
& + e^{-2\rho}\sinh\rho\cdot\cosh\rho(p^l_m\Gamma^m_{nj} - p^m_j\Gamma^l_{nm})
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\bar{R}^l_{0j0,0} = & -\sinh^3\rho\cdot E^l_{0j0,0} - 2\sinh^2\rho\cdot\cosh\rho E^l_{0j0} \\
& + (e^{-2\rho}\sinh\rho\cdot\cosh^2\rho + e^{-2\rho}\sinh^3\rho - 2e^{-3\rho}\sinh^2\rho\cdot\cosh\rho)p^l_j \\
& + e^{-2\rho}\sinh^2\rho\cdot\cosh\rho\cdot\partial_\rho p^l_j + e^{-2\rho}\sinh\rho\delta^l_j
\end{aligned} \tag{19}
$$

Combine above calculations with (11), (13) and assumption (2), we arrive at

$$
\|\bar{\nabla}Rm(\bar{g})\|_{\bar{g}} \le \Lambda e^{(3-b)\rho}.
$$

Thus we complete the proof of the proposition. □

Consequently we have

Proposition 2.3. *Suppose* $(X^{n+1},\ g)$ *is a complete Riemannain manifold with an essential set* $\mathbf{E}$*, and that it satisfies the curvature condition (1) with* $a > 0$*. Then*

(1) *Near the boundary* $\tau = 0$*, if* $a > 2$*, then* $|\frac{\partial \bar{g}_{ij}}{\partial \tau}| \leq C\tau$*; if* $a = 2$*, then* $|\frac{\partial \bar{g}_{ij}}{\partial \tau}| \leq C|\tau \log \tau|$*; if* $0 < a < 2$*, then* $|\frac{\partial \bar{g}_{ij}}{\partial \tau}| \leq C\tau^{\alpha - 1}$*.*
(2) *Near the boundary* $\tau = 0$*, if* $a > 2$*, then* $|\frac{\partial^2 \bar{g}_{ij}}{\partial \tau^2}| \leq C$*; if* $a = 2$*, then* $|\frac{\partial^2 \bar{g}_{ij}}{\partial \tau^2}| \leq C|\log \tau|$*; if* $0 < a < 2$*, then* $|\frac{\partial^2 \bar{g}_{ij}}{\partial \tau^2}| \leq C\tau^{a-2}$*.*
(3) $\bar{g}_{ij}(\tau, \theta)$ *is lipschitz up to the boundary* $\tau = 0$*, if the condition (2) holds with* $k = 1$ *and* $b > 1$*.*
(4) *Near the boundary* $\tau = 0$*,* $|\frac{\partial^2 \bar{g}_{ij}}{\partial \tau \partial \theta^l}| \leq C\tau^{b-2}$*, if the curvature condition (2) with* $k = 1$ *and* $b > 1$*.*

Proof. For (1), recall that

$$\partial_\tau \bar{g}_{ij} = 2e^{-\rho}\bar{g}_{ij} - 2\frac{e^{-2\rho}}{\sinh \rho} p_{ij}.$$

Hence it is easily seen that (1) holds in the light of (8), (9), (10) and the fact that

$$\tau = \log(1 + \frac{2}{e^\rho - 1}) \sim e^{-\rho}, \quad \text{as } \rho \to \infty.$$

As for (2) we calculate

$$\frac{\partial^2 \bar{g}_{ij}}{\partial \tau^2} = 2(e^{-\rho}\partial_\tau \bar{g}_{ij} + e^{-\rho}\sinh \rho \bar{g}_{ij}) - 2(2e^{-2\rho}p_{ij} + e^{-2\rho} coth\rho p_{ij} - e^{-2\rho}\partial_\rho p_{ij})$$

Hence (2) in this proposition holds in the light of (11). (3) in this proposition was proved in [2]. Therefore we have only (4) in this proposition left to be proven. Again we calculate

$$\partial_\tau \partial_{\theta^l} \bar{g}_{ij} = 2e^{-\rho}\partial_{\theta^l}\bar{g}_{ij} - 2\frac{e^{-2\rho}}{\sinh \rho}\partial_{\theta^l} p_{ij}.$$

Thus (4) in this proposition is proven due to (13). □

So far we only employed ODE analysis and Riccati's equations to derive estimates on the compactified metric $\bar{g}$. To end this section we include a simple fact of calculus for later use.

Lemma 2.4. *Suppose* $f(x, y)$ *is a function on* $\mathbf{R}^{n-1} \times [0, +\infty)$ *and that*

$$|\nabla f| \leq Cy^{-\delta},$$

for some $\delta \in (0,1)$ and a positive constant C. Then there is a constant Λ that depends only on C such that

$$\|f\|_{C^{0,1-\delta}(\mathbf{R}^{n-1}\times[0,+\infty))} \leq \Lambda.$$

Proof. For any two points $(x_1,y_1),(x_2,y_2) \in \mathbf{R}^{n-1}\times[0,+\infty)$ with $y_2 \geq y_1$, we consider the following two cases:

Case 1: Suppose $y_2 \geq |x_1 - x_2|$, then we have

$$\begin{aligned} |f(x_1,y_1) - f(x_2,y_2)| &\leq |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)| \\ &\leq \frac{C}{1-\delta}(y_2^{1-\delta} - y_1^{1-\delta}) + Cy_2^{-\delta}|x_1 - x_2| \\ &\leq \frac{C}{1-\delta}|y_1 - y_2|^{1-\delta} + C|x_1 - x_2|^{1-\delta}, \end{aligned} \tag{20}$$

hence in this case the lemma is proven.

Case 2: Suppose $y_2 \leq |x_1 - x_2|$, in this case, take $y_0 = |x_2 - x_1|$, then we have

$$\begin{aligned} |f(x_1,y_1) - f(x_2,y_2)| &\leq |f(x_1,y_1) - f(x_1,y_0)| + |f(x_1,y_0) - f(x_2,y_0)| \\ &\quad + |f(x_2,y_0) - f(x_2,y_2)| \\ &\leq \frac{C}{1-\delta}(y_0^{1-\delta} - y_1^{1-\delta}) + \frac{C}{1-\delta}(y_0^{1-\delta} - y_2^{1-\delta}) + Cy_0^{-\delta}|x_1 - x_2| \\ &\leq \frac{C}{1-\delta}y_0^{1-\delta} + C|x_1 - x_2|^{1-\delta} \\ &\leq \Lambda|x_1 - x_2|^{1-\delta} \end{aligned} \tag{21}$$

Thus, we see in both cases Lemma is true. □

3. Harmonic coordinates at infinity

From the previous section we know that the normal derivatives of the compactified metric $\bar{g}$ is well under control when the curvature condition (1) holds with reasonably large $a > 0$, and so is the curvature of $\bar{g}$. But to use only ODE analysis to control the metric $\bar{g}$, even just in $C^{0,\alpha}$ norm, one needs to assume curvature condition (2) (cf. [2] and [3]). In this section we will make use of elliptic PDE to improve the basic estimates established in the previous section. Our approach is very straightforward. We want to construct a harmonic coordinate system near by the infinity to translate the estimates of curvature to the better regularity of the metric $\bar{g}$. Particularly when $a > 1$ the boundary $\mathbf{B}^n = \partial X^{n+1}$ is totally geodesic with respect to the compactified metric $\bar{g}$. Recall

$$\bar{g} = d\tau^2 + \bar{g}_{ij}(\tau,\theta)d\theta^i d\theta^j$$

in a local coordinate near by the infinity, $(0,\epsilon]\times\mathbf{B}$ for some small ϵ, as we chose before. Therefore we can build a double $N=[-\epsilon,\epsilon]\times\mathbf{B}$ of the manifold $(0,\epsilon]\times\mathbf{B}$ and extend the metric $\bar{g}$ to the double N evenly. It is clear that the doubled metric $\bar{g}$ is lipschitz when the curvature condition (1) holds with $a>0$ and the curvature condition (2) holds with $k=1$ and $b>1$. We now start to construct a harmonic coordinate in the double N across the boundary $\{0\}\times\mathbf{B}$.

Lemma 3.1. *Suppose that $(X^{n+1},\ g)$ is a complete Riemanaian manifold with an essential set $\mathbf{E}$. Suppose that the curvature condition (1) holds with $a>0$ and the curvature condition (2) holds with $k=1$ and $b>1$. Let $(\tau=\theta^0,\theta^1,\theta^2,\cdots\theta^n)$ be a local coordinate system out of the product structure near the infinity of X as before. Then there exists a constant C independent of τ and θ such that on a subset $[-\tau_0,\tau_0]\times\mathcal{O}$ of the double N, we have*

$$|\bar{\Delta}\theta^\alpha|\le C,\quad \forall\ \alpha=0,1,\cdots,n,$$

where $\mathcal{O}$ is an open set in $\mathbf{B}$, τ_0 is some small number and $\bar{\Delta}$ is the Laplacian operator with respect to the metric $\bar{g}$.

Proof. This is a simply consequence of the fact that the metric $\bar{g}$ is Lipschitz when the curvature condition (1) holds with $a>0$ and the curvature condition (2) holds with $k=1$ and $b>1$ by the work in [2] and [3]. □

Let $\phi=(\theta^0,\theta^1,\cdots,\theta^n)$ and $D_{3\tau_1}\subset(-\tau_0,\tau_0)\times\mathcal{O}$ be a geodesic ball in the double N centered on the boundary $\{0\}\times\mathbf{B}$, we consider the following Dirichlet problem:

$$\begin{cases}\bar{\Delta}\psi=0, & \text{in}\quad D_{3\tau_1},\\ \psi|_{\partial D_{3\tau_1}}=\phi|_{\partial D_{3\tau_1}},\end{cases}\tag{22}$$

Let $\xi=\psi-\phi$ and $\theta=\tau_1 z$. Then we have

$$\begin{cases}|\ \bar{\Delta}_z\xi\ |\le C\tau_1^2 & \text{in}\quad D_3\\ \xi|_{\partial D_3}=0.\end{cases}\tag{23}$$

Hence by $W^{2,p}$ interior estimates we obtain

$$\|\xi\|_{W^{2,p}(D_2)}\le C\tau_1^2,$$

where $p>1$ is arbitrary and C depends on n, p and the constant of lemma 3.1. Due to the Sobolev embedding theorem we have

$$\|\xi\|_{C^{1,\mu}(D_2)}\le C\tau_1^2,\ if p>n+1,$$

here $\mu=1-\frac{n}{p}$. By rescaling back to x-variable,

$$\|\xi\|_{C^{1,\mu}(D_{2\tau_1})}\le C\tau_1^{1-\mu},\ if p>n+1,$$

here C is a constant independent of τ, τ_1 and θ. Therefore, by choosing τ_1 sufficiently small we see ψ is harmonic coordinates in $D_{2\tau_1}$ and there exists a constant $\delta_0 > 0$ independent of τ, τ_1, and θ such that $|\det(D\psi)| \geq \delta_0$ for all point in $D_{2\tau_1}$.

Let $(D_{2\tau_1}, y^\gamma)$, $0 \leq \gamma \leq n$, be the harmonic coordinates we just constructed, in the following, we try to get some higher order smoothness for those coordinates near the boundary. First we take the advantage of (4) in Proposition 2.3. Namely,

Lemma 3.2. *Suppose that $(y^0, y^1, \cdots, y^n)$ is the harmonic coordinates constructed above. Then*

$$\|\frac{\partial}{\partial \tau} y^\gamma\|_{C^{1,\mu}(D_{\frac{3}{2}\tau_1})} \leq C,$$

where $\mu = 1 - \frac{n+1}{p}$, C is a constant, and p satisfies

(1) *$p > n+1$ if the curvature condition (1) holds with $a > 0$ and the curvature condition (2) holds with $k = 1$ and $b \geq 2$;*

(2) *$p \in (n+1, \frac{1}{2-b})$ if the curvature condition (1) holds with $a > 0$ and the curvature condition (2) holds with $k = 1$ and $2 - \frac{1}{n+1} < b < 2$.*

Moreover the harmonic coordinate functions y^γ are all even with respect to the variable τ.

Proof. Recall, in local coordinate ϕ

$$\bar{g} = d\tau^2 + \bar{g}_{ij} d\theta^i d\theta^j, \ 1 \leq i, j \leq n.$$

Let

$$\delta_h y = \frac{y(\tau + h, \theta) - y(\tau, \theta)}{h},$$

for $(\tau, \theta^1, \cdots, \theta^n) \in D_{\frac{3}{2}\tau_1}$ and $|h| < \frac{\tau_1}{4}$. Then

$$\delta_h \bar{\Delta} y = 0,$$

in $D_{\frac{3}{2}\tau_1}$. From the fact that

$$\|y\|_{W^{2,p}(D_{2\tau_1})} \leq C,$$

and

$$\frac{\partial^2 \bar{g}_{ij}}{\partial \tau \partial x} \in L^p,$$

due to (4) in Proposition 2.3 when $(b-2)p > -1$, we get

$$\|\bar{g}^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \delta_h y\|_{L^p(D_{\frac{3}{2}\tau_0})} \leq C,$$

which implies

$$\|\delta_h y\|_{W^{2,p}(D_{\frac{3}{2}\tau_1})} \leq C.$$

Let h tend to zero. Then the Lemma follows from the Sobolev embedding theorem. And evenness of each harmonic coordinate function follows simply from the maximum principle. $\square$

As a corollary, we have

Corollary 3.3. *Suppose that the curvature condition (1) holds with $a > 0$ and the curvature condition (2) holds with $k = 1$ and $b > 2 - \frac{1}{n+1}$. And let $\hat{g}_{\alpha\beta}$ be the components of metric $\bar{g}$ in the above harmonic coordinates. Then $\frac{\partial}{\partial\tau}\hat{g}_{\alpha\beta} \in W^{1,p}(\mathcal{U})$ for $p \in (n+1, \frac{1}{2-b})$, where $\mathcal{U} \subseteq D_{\frac{3}{2}\tau_1}$. Moreover $\hat{g}_{\alpha\beta}$ is even with respect to τ and*

$$\frac{\partial}{\partial\tau}\hat{g}_{\alpha\beta} = 0 \quad \text{at } \tau = 0.$$

As we doubled the manifold and extend the metric to the doubled manifold N evenly, we have

$$\bar{R}m(\tau,\theta) = \begin{cases} \bar{R}m(\tau,\theta), & \tau > 0, \\ \bar{R}m(-\tau,\theta), & \tau < 0. \end{cases} \tag{24}$$

To finally utilize the curvature equations we need

Proposition 3.4. *Suppose that the curvature condition (1) holds with $a > 0$ and the curvature condition (2) holds with $k = 1$ and $b > 2 - \frac{1}{n+1}$. Then $\hat{g}_{\alpha\beta}$ is weak solution to the following equations*

$$\frac{1}{2}\bar{\Delta}\hat{g}_{\alpha\beta} + Q_{\alpha\beta}(\partial\hat{g}, \hat{g}) = -\hat{R}_{\alpha\beta},$$

in $\mathcal{U}$, where $\bar{\nabla}$ is gradient operator with respect to metric $\bar{g}$, $Q(\partial\hat{g}, \hat{g})$ is bilinear form of $\hat{g}$ and its first derivative, $d\bar{V}$ is the volume form with respect to $\bar{g}$, $\hat{R}_{\alpha\beta}$ is component of Ricci curvature tensor in $(\mathcal{U}, y^\gamma)$ excluding the boundary $\tau = 0$.

Proof. Let

$$\mathcal{U}_+ = \{(\tau,\theta) \in \mathcal{U} | \tau \geq 0\},$$

and

$$\mathcal{U}_- = \{(\tau,\theta) \in \mathcal{U} | \tau \leq 0\},$$

$$T = \mathcal{U}_+ \bigcap \mathcal{U}_-,$$

$\vec{n}_\pm$ be the outward unit normal vector of $\mathcal{U}_\pm$ on T. Since $\hat{g}$ is smooth in $\mathcal{U}$ except on T, we have, for each smooth function supported inside $\mathcal{U}$,

$$\frac{1}{2}\Big(\int_T \frac{\partial\hat{g}_{\alpha\beta}}{\partial\vec{n}_+}\cdot\eta d\bar{S} - \int_{\mathcal{U}_+} \bar{\nabla}\hat{g}_{\alpha\beta}\cdot\bar{\nabla}\eta d\bar{V}\Big) + \int_{\mathcal{U}_+} Q_{\alpha\beta}(\partial\hat{g}, \hat{g})\cdot\eta d\bar{V} = -\int_{\mathcal{U}_+} \hat{R}_{\alpha\beta}\cdot\eta d\bar{V},$$

and

$$\frac{1}{2}\left(\int_T \frac{\partial \hat{g}_{\alpha\beta}}{\partial \vec{n}_-}\cdot \eta d\bar{S} - \int_{\mathcal{U}_-} \bar{\nabla}\hat{g}_{\alpha\beta}\cdot\bar{\nabla}\eta d\bar{V}\right) + \int_{\mathcal{U}_-} Q_{\alpha\beta}(\partial\hat{g},\hat{g})\cdot\eta d\bar{V} = -\int_{\mathcal{U}_-} \hat{R}_{\alpha\beta}\cdot\eta d\bar{V}.$$

Here we used Proposition 2.2 and Corollary 3.3. Note that on T,

$$\frac{\partial}{\partial \vec{n}_-} = \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial \vec{n}_+} = -\frac{\partial}{\partial \tau}.$$

By Corollary 3.3, we hence obtain

$$\int_T \frac{\partial \hat{g}_{\alpha\beta}}{\partial \vec{n}_+}\cdot \eta d\bar{S} = \int_T \frac{\partial \hat{g}_{\alpha\beta}}{\partial \vec{n}_-}\cdot \eta d\bar{S} = 0,$$

which completes the proof of the Proposition. □

Now we are ready to use harmonic coordinates at the infinity to prove a regularity result as a step stone for the proof of Theorem 1.2..

Theorem 3.5. *Suppose that $(X^{n+1},\ g)$ is a complete Riemannain manifold with an essential set* **E** *and that it satisfies the curvature condition (1) with $a > 0$ and the curvature condition (2) with $k = 1$ and $b > 2 - \frac{1}{n+1}$. Then there is differentiable structure on $\bar{X}$, which is smooth in the interior of X, and $W^{3,p}$ up to the boundary, where p is some positive constant bigger than $n+1$. And in this differentiable structure, we have*

(1) *$\bar{g}$ is smooth in the interior of X and is $W^{2,p}$ up to the boundary for some $p > n+1$. In particular, it is $C^{1,\alpha}$ smooth up to the boundary, for some $\alpha \in (0,1)$;*
(2) *$\bar{g}$ is smooth in the interior of X and it is $W^{2,p}$ up to the boundary for any $p > n+1$ if in fact $b \geq 2$. In particular, it is $C^{1,\alpha}$ smooth up to the boundary, for any $\alpha \in (0,1)$.*

Proof of Theorem 3.5. By the definition and a direct computation, we see that $\hat{g}_{\alpha\beta} \in W^{1,p}(\mathcal{U}, y^\gamma)$, for some $p \in (n+1, \frac{1}{2-b})$, Then, by Proposition 3.4 and the standard L^p theory in PDE, we obtain that $\hat{g}_{\alpha\beta} \in W^{2,p}(\mathcal{U}, y^\gamma)$ and there is a constant C which depends on p such that

$$\|\hat{g}_{\alpha\beta}\|_{L^p(\mathcal{U})} + \sum_\gamma \|\frac{\partial}{\partial y^\gamma}\hat{g}_{\alpha\beta}\|_{L^p(\mathcal{U})} + \sum_{\gamma,\mu}\|\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\gamma \partial y^\mu}\|_{L^p(\mathcal{U})} \leq C.$$

Particularly $\hat{g}_{\alpha\beta} \in C^{1,\mu}(\mathcal{U}, y^\gamma)$ due to the standard Sobolev embedding theorem for some $\mu \in (0,1)$.

Now we have constructed the harmonic coordinates $(\mathcal{U}, y^\gamma)$ on N, and $\hat{g}_{\alpha\beta}$ has better regularity than $\bar{g}_{\alpha\beta}$. By taking $\mathcal{U}\bigcap \bar{M}$, which is still denoted by $\mathcal{U}$, we get a coordinates covering on M. Let $(\mathcal{U}, y^\gamma)$ and $(\mathcal{V}, z^\sigma)$ be two distinct

harmonic coordinates on $(N,\ \bar{g})$. And suppose $\mathcal{U}\bigcap\mathcal{V}$ is nonempty. Then, by the standard arguments in PDE, we see that there is a constant C with

$$\|z^\sigma\|_{W^{3,p}(\mathcal{U}\bigcap\mathcal{V},y^\gamma)} \leq C$$

Putting all these harmonic coordinates together we then obtain a differentiable structure on $\bar{X}^{n+1}$. Thus we finish to prove Theorem 3.5. $\square$

Proof of Theorem 1.2. Recall from Proposition 2.2

$$\|\bar{\nabla}Rm(\bar{g})\|_{\bar{g}} \leq \Lambda\tau^{-(3-b)}.$$

Hence by Lemma 2.4 and $b > 2$, we know $\bar{R}m \in C^{0,\mu_1}(\mathcal{U})$. On the other hand, $\partial\hat{g}_{\alpha\beta} \in C^{0,\mu_2}(\mathcal{U})$, due to Theorem 3.5. Therefore we may recall the equation

$$\frac{1}{2}\bar{\Delta}\hat{g}_{\alpha\beta} + Q_{\alpha\beta}(\partial\hat{g},\hat{g}) = -\hat{R}_{\alpha\beta},$$

where $Q_{\alpha\beta}(\partial\hat{g},\hat{g})$ and $\hat{R}_{\alpha\beta}$ are all Hölder continuous. Thus Theorem 1.2 follows from the standard Schauder theory in elliptic PDE. $\square$

4. Improvement by the Ricci flow

In this section we show that the Ricci flow can be used to help to get the regularity without assuming the curvature condition (2). Let $(X^{n+1},\ g)$ be a complete Riemannian manifold satisfying the curvature condition (1) with $a > 0$. We consider the Ricci flow equations

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}g_{\alpha\beta} = -2R_{\alpha\beta} \\ g_{\alpha\beta}(x,0) = g_{\alpha\beta}(x) \quad \text{on} \quad X. \end{array}\right. \tag{25}$$

We will also consider the normalized Ricci flow as follows:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}g_{\alpha\beta} = -2ng_{\alpha\beta} - 2R_{\alpha\beta}, \\ g_{\alpha\beta}(x,0) = g_{\alpha\beta}(x) \quad \text{on} \quad X, \end{array}\right. \tag{26}$$

We recall that, if g^N solves the normalized Ricci flow (26) and let

$$\left\{\begin{array}{l} \tau(t) = \frac{1}{2n}e^{2nt} - \frac{1}{2n}, \\ g^U_{\alpha\beta}(x,\tau) = (1+2n\tau)g^N_{\alpha\beta}(x,t), \end{array}\right. \tag{27}$$

then g^U solves the Ricci flow (25). Hence if we know one we know the other. Since our initial metric satisfies the curvature condition (1), by the works of Shi (see Theorem 1.1 in [16]), there exist constants $T, C_1, C_2 > 0$, which only depend on n and the constant in (1) such that the evolution equation (25) has a smooth solution $g(\cdot,t)$ for a short time $t \in [0,T]$, which satisfies the following estimates:

$$\sup_{x\in X}\|R_{\alpha\beta\gamma\delta}(x,t)\| \leq\ C_1,\ 0 \leq t \leq T$$

and

$$\sup_{x\in X}\|\nabla R_{\alpha\beta\gamma\delta}(x,t)\| \le \frac{C_2}{\sqrt{t}},\ \ 0\le t\le T.$$

Let

$$E_{\alpha\beta\gamma\delta}(g^U) := R_{\alpha\beta\gamma\delta}(g^U) + \frac{1}{1+2(n-1)t}(g^U_{\alpha\gamma}g^U_{\beta\delta} - g^U_{\alpha\delta}g^U_{\beta\gamma}).$$

Then

$$E_{\alpha\beta\gamma\delta}(g^N) = R_{\alpha\beta\gamma\delta}(g^N) + (g^N_{\alpha\gamma}g^N_{\beta\delta} - g^N_{\alpha\delta}g^N_{\beta\gamma}).$$

Notice that $\nabla E = \nabla Rm$. Direct computations from the evolution equations of Riemann curvature tensor give us the following:

Lemma 4.1. *Suppose that g solve the Ricci flow (25) and that $E = E(g)$. Then the evolution equations of $\|E\|^2$ and $\|\nabla E\|^2$ are given by*

$$\frac{\partial}{\partial t}\|E\|^2 = \Delta\|E\|^2 - 2\|\nabla E\|^2 + E*E*E + E*E$$

and

$$\frac{\partial}{\partial t}\|\nabla E\|^2 = \Delta\|\nabla E\|^2 - 2\|\nabla^2 E\|^2 + \nabla E*\nabla E + E*\nabla E*\nabla E,$$

where $E*E$ stands for terms that are contractions of E and E.

Now we will use the maximum principle to estimate $\|E\|$ and $\|\nabla E\|$.

Lemma 4.2. *Let $g(t)$ be the smooth solution of (25) and ρ_t the distance function corresponding to $g(t)$, $0\le t\le T$, then*

$$|\Delta_{g(t)}\rho_0| \le C,\ \ 0\le t\le T,$$

here C depends only on n and the constant in (1).

Proof. First we recall a fact that

$$\frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial t} = -g^{\gamma\delta}(R_{\alpha\delta,\beta} + R_{\beta\delta,\alpha} - R_{\alpha\beta,\delta}). \tag{28}$$

Then we calculate

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_{g(t)}\rho_0) &= \frac{\partial}{\partial t}(g^{\alpha\beta}(\nabla^2\rho_0)_{\alpha\beta})\\ &= 2g^{\alpha\gamma}g^{\beta\delta}R_{\gamma\delta}(\nabla^2\rho_0)_{\alpha\beta} + g^{\alpha\beta}g^{\gamma\delta}(R_{\beta\delta,\alpha} + R_{\alpha\delta,\beta} - R_{\alpha\beta,\delta})\partial_\gamma\rho_0.\end{aligned}$$

Due to Bianchi identity

$$g^{\alpha\beta}(R_{\beta\delta,\alpha} + R_{\alpha\delta,\beta} - R_{\alpha\beta,\delta}) = 0$$

we arrive at

$$\begin{aligned}&\frac{\partial}{\partial t}(\Delta_{g(t)}\rho_0) = 2g^{\alpha\gamma}g^{\beta\delta}R_{\gamma\delta}(\nabla_0^2\rho_0)_{\alpha\beta}\\ &=2g^{\alpha\gamma}g^{\beta\delta}R_{\gamma\delta}(\nabla_0^2\rho_0)_{\alpha\beta} + 2g^{\alpha\gamma}g^{\beta\delta}g^{\gamma\sigma}R_{\gamma\delta}(R_{\beta\sigma,\alpha} + R_{\alpha\sigma,\beta} - R_{\alpha\beta,\sigma})(s)\partial_\gamma\rho_0 \cdot t,\end{aligned}$$

where $0 < s < t \leq T$, $\nabla_0^2\rho_0$ is Hessian of ρ_0 with respect to metric g_0. This lemma then follows from Shi's work [16]. □

Now let μ, ν and η be constants which are to be determined later. By we calculate from Lemma 4.2 that

$$\begin{aligned}&\frac{\partial}{\partial t}(e^{\mu\rho_0}\|E\|^2) \leq \Delta(e^{\mu\rho_0}\|E\|^2)\\ &+ < 2\mu\nabla\rho_0, \nabla(e^{\mu\rho_0}\|E\|^2) > + C(e^{\mu\rho_0}\|E\|^2) - 2e^{\mu\rho_0}\|\nabla E\|^2,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(e^{\nu\rho_0}\|\nabla E\|^2) \leq \Delta(e^{\nu\rho_0}\|\nabla E\|^2) &+ < 2\nu\nabla\rho_0, \nabla(e^{\nu\rho_0}\|\nabla E\|^2) >\\ &+ C(e^{\nu\rho_0}\|\nabla E\|^2) - 2e^{\nu\rho_0}\|\nabla^2 E\|^2,\end{aligned}$$

where C is large enough and independent of t. Therefore

$$\begin{aligned}\frac{\partial}{\partial t}(e^{\mu\rho_0}\|E\|^2 + \eta te^{\nu\rho_0}\|\nabla E\|^2) &\leq \Delta(e^{\mu\rho_0}\|E\|^2 + \eta te^{\nu\rho_0}\|\nabla E\|^2)\\ &+C(e^{\mu\rho_0}\|E\|^2 + \eta te^{\nu\rho_0}\|\nabla E\|^2) + (\eta e^{\nu\rho_0} - 2e^{\mu\rho_0})\|\nabla E\|^2\\ &\leq \Delta(e^{\mu\rho_0}\|E\|^2 + \eta te^{\nu\rho_0}\|\nabla E\|^2)\\ &+ C(e^{\mu\rho_0}\|E\|^2 + \eta te^{\nu\rho_0}\|\nabla E\|^2),\end{aligned} \tag{29}$$

if we choose constants $\mu = \nu$ and $\eta \leq 2$.

Thus we have:

Proposition 4.3. *Suppose $(X^{n+1},\ g)$ is a complete Riemannian manifold satisfying the curvature condition (1) with $a > 0$. Let $(M, g(t))$, $t \in [0, T]$, be a complete solution of the normalized Ricci flow (26). Then there exists constants T_0 and C such that*

$$\|E(g(t))\| \leq Ce^{-a\rho_0}$$

and

$$\|\nabla E(g(t))\| \leq \frac{C}{\sqrt{t}}e^{-a\rho_0},$$

where ∇ is with respect to metric $g(t)$, and C is independent of t, $0 < t \leq T_0 \leq T$.

Proof. It suffices to prove the same conclusion for unnormalize Ricci flow (25). From the analysis above, choose T smaller if necessary, we know $(X,\ g(t))$, $t \in [0,T]$, be a complete solution of the Ricci flow (25) with uniformly bounded curvature. In fact, due to (1) on $(X,\ g(0))$,

$$e^{2a\rho_0}\|E\|^2(x,0) \leq C.$$

So $u(x,t) = e^{2a\rho_0}\|E\|^2 + te^{2a\rho_0}\|\nabla E\|^2 - C$ is a weak subsolution of the heat equation $(\frac{\partial}{\partial t} - \triangle)u(x,t) = 0$ on $X \times [0,T]$ with $u(x,0) = e^{2a\rho_0}\|E\|^2 - C \leq 0$. On the other hand, we can choose sufficiently large ω such that

$$\int_0^{T_1}\int_M exp(-\omega d^2_{g_0}(x,o))u^2_+(x,s)dV_{g(t)}dt < \infty,$$

where $d_{g_0}(x,o)$ is a distance function to a fixed point $o \in M$ with respect to g_0. Then according to the results of Krap and Li [13], $u(x,t) \leq 0$ for all $(x,t) \in X \times [0,T]$. Then

$$e^{2a\rho_0}\|E\|^2 - C \leq 0$$

and

$$\eta te^{\mu\rho_0}\|\nabla E\|^2 - C \leq 0.$$

Hence the proof is complete. □

So far the normalized Ricci flow $g(x,t)$ preserves the asymptotical curvature behavior of the initial metric. Next we want to show that there is a compact set $\mathbf{E} \subset X$ such that it is essential set for all $g(x,t)$, $t \in [0,T]$, if $\mathbf{E} \subset X$ is an essential set for the initial metric, at least for a sufficiently small T. To do that, according to Theorem 4 in [4], it suffices to show that $\mathbf{E}$ is strictly convex. Since, in the light of Proposition 4.3, we may choose $\mathbf{E}$ large enough so that sectional curvature of $g(x,t)$ is negative out of $\mathbf{E}$ for $t \in [0,T]$.

Proposition 4.4. *Suppose $(X^{n+1},\ g)$ is a complete Riemannian manifold with an essential set $\mathbf{E}$ and that it satisfies the curvature condition (1) with $a > 0$. Let ρ be the distance to $\partial\mathbf{E}$ in $(X^{n+1},\ g)$. And suppose that $g(\cdot,t)$ is the solution to the normalized Ricci flow equations (26). Then there is $T > 0$ and Λ_0 such that*

$$\mathbf{E}_1 = \{x \in X : \rho \leq 2\Lambda_0\}$$

is strictly convex with respect to $g(\cdot,t)$ for all $t \in [0,T]$. Therefore $\mathbf{E}_1$ is an essential set in X for all $g(\cdot,t)$ with $t \in [0,T]$.

Proof. First we claim that there is $T > 0$ and Λ_0 such that

$$\nabla^2_{g(\cdot,t)} \cosh\rho \geq \frac{\sinh\rho}{4}g(\cdot,t),$$

for all $t \in [0,T]$ and $\rho \geq \Lambda_0$. Let $\frac{\partial}{\partial \theta^0} = \frac{\partial}{\partial \rho}$. Then

$$\nabla^2_{g_0} \cosh \rho (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}) = \cosh \rho,$$

$$\nabla^2_{g_0} \cosh \rho (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta^i}) = 0,$$

and

$$\nabla^2_{g_0} \cosh \rho (\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}) = -\Gamma^1_{ij} \sinh \rho = S^k_i g_{kj}(\rho, \theta) \sinh \rho.$$

We then can choose Λ_0, which is independent of t, such that for all $\rho \geq \Lambda_0$,

$$S^k_i g_{kj}(\rho, \theta) \sinh \rho \geq \frac{1}{2} \sinh \rho \cdot g_{ij}(\rho, \theta),$$

in the light of (8), (9) and (10). Therefore

$$\nabla^2_{g_0} \cosh \rho \geq \frac{1}{2} \sinh \rho \cdot g_0$$

At $t = 0$. On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla_i \nabla_j \cosh \rho) &= \sinh \rho g^{0l}(R_{jl,i} + R_{il,j} - R_{ij,l}) \\ &= \sinh \rho (R_{j0,i} + R_{i0,j} - R_{ij,0}), \end{aligned}$$

which implies

$$|\frac{\partial}{\partial t}(\nabla_i \nabla_j \cosh \rho)| \leq C \sinh \rho ||\nabla Ric|| \leq C t^{-\frac{1}{2}} e^{-\alpha \rho_0} \sinh \rho.$$

Choose Λ_0 large enough and time interval small enough if necessary, we hence proved the first claim.

To see that $\mathbf{E}_1$ is strictly convex in all $(X, g(t))$. Let $f = \cosh \rho$. For any p, $q \in \mathbf{E}_1$, then $f(p) \leq \cosh(2\Lambda_0)$, $f(q) \leq \cosh(2\Lambda_0)$. Let γ_t be any geodesic joining p and q in the metric $g(\cdot, t)$. Then

$$f(\gamma_t(s))'' > 0, \quad \text{for all } s \in (0,1),$$

from which it is easily seen that

$$f(\gamma_t(s)) < \cosh(2\Lambda_0),$$

for all $s \in (0,1)$. Thus we construct an essential set $\mathbf{E}$ of $(M,\ g(t))$, for all $t \in [0,T]$. □

By now we may use the regularity results in the previous section and regularity theorem of [3] to the metric $g(\cdot, t)$ for $t \in (0, T]$. First we notice that

Proposition 4.5. *Suppose $(X^{n+1},\ g)$ is a complete Riemannian manifold with an essential set* $\mathbf{E}$ *and that it satisfies the curvature condition (1) with $a>0$. Let $g(\cdot,t)$ be the solution to the normalized Ricci flow (26) and that* $\mathbf{E}$ *is an essential set for all $g(\cdot,t)$. Let $\bar{g}_{ij}(\cdot,t)=\sinh^{-2}\rho_t g(\cdot,t)$, where ρ_t is the distance function to* $\mathbf{E}$ *with respect to the metric $g(\cdot,t)$. Then there are constants T and C such that*

$$\|\bar{g}(\cdot,t)-\bar{g}(\cdot,0)\|\le Ct$$

for all $t\in[0,T]$.

Proof. According to the equations

$$\frac{\partial}{\partial t}g_{\alpha\beta}=-2ng_{\alpha\beta}-2R_{\alpha\beta},$$

we easily see that

$$\|g(\cdot,t)-g(\cdot,0)\|\le Ct,$$

where C is independent of t.

Let $\rho(\cdot,t)$ be the distance function to $B=\partial\mathbf{E}$ with respect to $g(\cdot,t)$. For any $x\in M\backslash\mathbf{E}$, suppose that $p\in\mathbf{B}$ is the point such that the length of the geodesic joining x and p is just the distance of x to $\mathbf{B}$ with respect to $g(\cdot,0)$. Denote this geodesic by $\gamma(s)$, where $s\in[0,1],\ \gamma(0)=p,\ \gamma(1)=x,\ \rho(x,0)$ $=L(\gamma,0)$ and $L(\gamma,t)$ denotes the length of γ with respect to $g(\cdot,t)$. We know that $g(\cdot,t)$ and $g(\cdot,0)$ are uniformly quasi-isometric when $t\in[0,T]$. Hence there exists a constant λ, independent of t, such that

$$\lambda^{-1}L(\gamma,0)\le L(\gamma,t)\le\lambda L(\gamma,0),\ \text{for any}\ t\in[0,T].$$

On the other hand we may compute

$$\begin{aligned}\frac{\partial}{\partial t}L(\gamma,t)&=\frac{\partial}{\partial t}\int_0^1(g_{\alpha\beta}(\gamma,t)\dot\gamma^\alpha\dot\gamma^\beta)^{\frac12}ds\\&=\int_0^1(g_{\alpha\beta}(\gamma,t)\dot\gamma^\alpha\dot\gamma^\beta)^{-\frac12}(-ng_{\mu\nu}(\gamma,t)-R_{\mu\nu}(\gamma,t))\dot\gamma^\mu\dot\gamma^\nu ds\\&=-\int_0^1(g_{\alpha\beta}(\gamma,t)\dot\gamma^\alpha\dot\gamma^\beta)^{-\frac12}E_{\mu\nu}(\gamma,t)\dot\gamma^\mu\dot\gamma^\nu ds,\end{aligned}$$

where $E_{\alpha\beta}=g^{\gamma\delta}(t)E_{\alpha\gamma\beta\delta}(g(\cdot,t))$, and

$$E_{\alpha\beta\gamma\delta}(g)=R_{\alpha\beta\gamma\delta}(g)+(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}).$$

Hence according to Proposition 4.3 we have

$$\begin{aligned}\int_0^1 (g_{ij}(\gamma,t)\dot\gamma^i\dot\gamma^j)^{-\frac{1}{2}}E_{ij}(\gamma,t)\dot\gamma^i\dot\gamma^j ds &\le \int_0^1 Ce^{-\alpha sL_0}\lambda^{\frac{1}{2}}(g_{ij}(\gamma,0)\dot\gamma^i\dot\gamma^j)^{\frac{1}{2}}ds\\ &\le C\lambda^{\frac{1}{2}}L_0\int_0^1 e^{-asL_0}ds\\ &\le C\lambda^{\frac{1}{2}}a^{-1},\end{aligned}$$

which implies

$$|L(\gamma,t)-L_0| \le \frac{C\lambda^{\frac{1}{2}}}{a}t, \text{ for any } t\in[0,T],$$

and

$$\rho(x,t)\le L(\gamma,t)\le L_0+\frac{C\lambda^{\frac{1}{2}}}{a}t=\rho(x,0)+\frac{C\lambda^{\frac{1}{2}}}{a}t.$$

Similarly we get

$$\rho(x,0)=L_0\le\rho(x,t)+\frac{C\lambda^{\frac{1}{2}}}{a}t.$$

Therefore

$$|\rho(x,t)-\rho(x,0)|\le\frac{C\lambda^{\frac{1}{2}}}{a}t,$$

for any $t\in[0,T]$. Thus

$$\|\bar g(x,t)-\bar g(x,0)\|\le C(\lambda,a)t,$$

for all $x\in M\setminus\mathbf{E}$, and all $t\in[0,T]$, which completes the proof of the proposition. $\square$

As another easy consequence of the use of the help from the Ricci flow we now give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $g(x,t)$ be the metric constructed by the normalized Ricci flow (26). Then, according to Theorem A in [3] and Proposition 4.3 in the above, we have

$$\|\bar g(x,t)\|_{C^{0,a}}\le\Lambda t^{-\frac{1}{2}},$$

on $\bar X$. Recall from the proof of Proposition 4.5

$$|\bar g(x,t)-\bar g(x,0)|\le Ct.$$

Hence, if let $\mu = \frac{2}{3}a$, we have, for any $t > 0$,

$$
\begin{aligned}
\frac{\|\bar{g}(x,0)-\bar{g}(y,0)\|}{|x-y|^{\mu}} &\leq \frac{\|\bar{g}(x,t)-\bar{g}(x,0)\|}{|x-y|^{\mu}} + \frac{\|\bar{g}(y,t)-\bar{g}(y,0)\|}{|x-y|^{\mu}} \\
&\quad + \frac{\|\bar{g}(x,t)-\bar{g}(y,t)\|}{|x-y|^{\mu}} \\
&\leq Ct\cdot|x-y|^{-\mu} + \Lambda t^{-\frac{1}{2}}\cdot|x-y|^{a-\mu}
\end{aligned} \tag{30}
$$

Take $t = |x-y|^{\frac{2}{3}a}$, we obtain

$$\frac{\|\bar{g}(x,0)-\bar{g}(y,0)\|}{|x-y|^{\mu}} \leq C.$$

Thus we finish the proof. □

Next we would like to drop the curvature condition (2) in Theorem 3.5 with the help from the Ricci flow method indicated in the above. By Theorem 3.5, if we denote $\{x^{\gamma}\}$ to be harmonic coordinates with respect to the metric $\bar{g}(\cdot,t)$, then $\bar{g}_{\alpha\beta}(x,t)$ is smooth in the interior of X and it is $W^{2,p}$ up to the boundary, in the differential structure associated with the harmonic coordinates. For a technical reason we first smoothen each $\bar{g}(\cdot,t)$ in order to allow us to apply Theorem 5.4 in [12] (page 187). We continue to use the notations in the proof of Theorem 3.5 in the previous section.

Lemma 4.6. *Suppose $\bar{g}_{\alpha\beta}(\cdot,t)$ is in $W^{2,p}$ for some $p > \frac{n+1}{2}$ in a coordinate neighborhood $\mathcal{U}$ in the doubled manifold N. Let $\bar{g}^{\epsilon}_{\alpha\beta}(\cdot,t)$ be the ϵ-mollification of $\bar{g}_{\alpha\beta}(\cdot,t)$, i.e. let $\mu(x)$ is a smooth function with compact support inside $\mathcal{U}$ such that $\int\mu(x)dx = 1$ and let $\mu_{\epsilon}(x) = \frac{1}{\epsilon^{n+1}}\mu(\frac{x}{\epsilon})$, then*

$$\bar{g}^{\epsilon}_{\alpha\beta}(x,t) = \int_M \bar{g}_{\alpha\beta}(y,t)\mu_{\epsilon}(x-y)dy.$$

Then we have

1. *$\bar{g}^{\epsilon}_{\alpha\beta}(\cdot,t)$ is C^{∞} smooth;*
2. *$\|Rm(\bar{g}^{\epsilon})\|_{L^p} \leq C$, where C is independent of k;*
3. *There is $\epsilon_k \to 0$ so that $\bar{g}^{\epsilon_k}_{\alpha\beta}(\cdot,\epsilon_k)$ converges to $\bar{g}_{\alpha\beta}(\cdot,0)$ uniformly in any compact subset of $\mathcal{U}$.*

Proof. We only need to prove the last two statements. For convenience we sometime drop the indices for the metrics if there is no confusion. Note that $W^{1,p} \subset L^{2p}$ since $p > \frac{n+1}{2}$, we hence have

$$\|(Rm(\bar{g}^{\epsilon}))\|_{L^p} \leq C\|\bar{g}\|_{W^{2,p}}.$$

Now let us prove the third statement. Clearly we have

$$|\bar{g}^{\epsilon}_{\alpha\beta}(\cdot,t) - \bar{g}_{\alpha\beta}(\cdot,0)| \leq |\bar{g}^{\epsilon}_{\alpha\beta}(\cdot,t) - \bar{g}_{\alpha\beta}(\cdot,t)| + |\bar{g}_{\alpha\beta}(\cdot,t) - \bar{g}_{\alpha\beta}(\cdot,0)|.$$

For the second term, by Proposition 4.5, we know

$$\lim_{t\to 0}\|\bar{g}(\cdot,t)-\bar{g}(\cdot,0))\|=0.$$

As for the first term, since $p>\frac{n+1}{2}$, we know $\bar{g}$ is continuous due to the assumptions. Therefore it is rather easy to see that

$$\lim_{\epsilon\to 0}\|\bar{g}^{\epsilon}-\bar{g}\|=0.$$

Thus we finish to prove the Proposition. □

For simplicity we denote $\bar{g}^{\epsilon_k}(\cdot,\epsilon_k)$ as $\bar{g}^k$ in the following.

Lemma 4.7. *Suppose $(X^{n+1},\ g)$ is a complete manifold with an essential set and that it satisfies the curvature condition (1) with $a>2-\frac{2}{n+1}$. Let $\mathcal{U}$ be the coordinate neighborhood in the doubled manifold where $\bar{g}^k$ was constructed in Lemma 4.6. Then for any point on T it admits a neighborhood $\mathcal{V}\subset\mathcal{U}$ so that*

(1) *for each $\bar{g}^k$ there is harmonic coordinates $H_k=(y_k^0,y_k^1,\cdots,y_k^n)$ on $\mathcal{V}$;*
(2) *there is a positive constant δ_0, which is independent of k, such that $\det(dH_k)\geq\delta_0>0$ on $\mathcal{V}$;*
(3) *$\|\hat{g}^k_{\alpha\beta}\|_{W^{2,p}(\mathcal{V})}\leq C$, for some $p>\frac{n+1}{2}$, and $\hat{g}^k_{\alpha\beta}$ is components of $\bar{g}^k$ under harmonic coordinates $(\mathcal{V},y_k^{\gamma})$, and C is constant that is independent of k.*
(4) *If $a\geq 2$, then (3) is true for any $p>1$.*

Proof. Indeed, due to Proposition 2.2 and Lemma 4.6, we know that

$$\|Rm(\bar{g}^k)\|_{L^p}\leq C.$$

Since $\bar{g}^k$ converges to $\bar{g}$ uniformly, we see that for any q in $\bar{X}$ and $s\leq 1$, we have $Vol(B(q,s))\geq\Lambda s^n$, where $B(q,s)$ is geodesic ball with radius s and center q and Λ is independent of s. Hence, by Theorem 5.4 in [12] (P.187), we obtain the existence of the neighborhood $\mathcal{V}$ and therefore the Lemma is proven. □

Notice that we have proved that $\bar{g}^k_{\alpha\beta}$ converges to $\bar{g}_{\alpha\beta}$ uniformly and that

$$\|\hat{g}^k_{\alpha\beta}\|_{W^{2,p}(\mathcal{V})}\leq C$$

where C is independent of k and $p>n+1$ when $a>2-\frac{1}{n+1}$. Because each y_k^{γ} is harmonic with respect to metric $\bar{g}^k$, from the standard theory of elliptic PDE we have

$$||y_k^{\gamma}||_{C^{2,\mu}(\mathcal{V})}\leq C,$$

for some $\mu>0$. Now we are well prepared to prove Theorem1.5.

Proof of Theorem 1.5. The a priori estimates from the standard theory of elliptic PDE are always available, but the missing key point is to find a fixed common domain $\mathcal{V}$ for the family of elliptic PDE's and their solutions, which has been shown using Theorem 5.4 in [12] (P.187) in the proof of Lemma 4.7. Indeed, by Theorem 3.5 and Lemma 4.7, for each k, we have differential structure $\Gamma_k = \{(\mathcal{V}, y_k^\gamma)\}$ and the components $\hat{g}^k_{\alpha\beta}$ for $\bar{g}^k$ in the harmonic coordinates $\{y_k^\gamma\}$. Then, taking a subsequence if necessary, we may assume

(1) y_k^γ converges to y^γ in $C^{2,\mu'}(\mathcal{V}')$ for some $\mu' < \mu$ and $\mathcal{V}' \subset\subset \mathcal{V}$ and;
(2) $\hat{g}^k_{\alpha\beta}$ converges weakly to $\hat{g}_{\alpha\beta}$ in $W^{2,p}(\mathcal{V})$,

where clearly $\hat{g}_{\alpha\beta}$ is the components of the metric $\bar{g}(\cdot, 0)$ in the coordinates y^γ if $\{y^\gamma\}$ is indeed a coordinate system, which is readily seen because $\{y_k^\gamma\}$ are coordinate systems.

Finally, suppose $(\mathcal{U}, y^\gamma)$, $(\mathcal{V}, z^\gamma) \in \Gamma$, then z^γ is harmonic in $\mathcal{U} \bigcap \mathcal{V}$ with respect to metric $\bar{g}$, and $\bar{g}$ is $W^{2,p}$-smooth under harmonic coordinates ($\mathcal{U} \bigcap \mathcal{V}$, y^γ), thus, we get

$$\|\frac{\partial z^\gamma}{\partial y^\nu}\|_{W^{2,p}(\mathcal{U}\bigcap\mathcal{V})} \leq C,$$

where C is a constant depends only Γ and p. Thus we finish to prove the Theorem 1.5. $\square$

5. Rigidity Theorems

About rigidity in this context there are three different approaches given in [1], [15] and [17] (see also [10]) respectively. In [1] the manifolds are assumed to be spin and the regularity of the conformal compactification is assumed to be very high. In [15] it still assumes the regularity of order $C^{3,\alpha}$, even though no spin condition is assumed for $n \leq 7$. In [17] it takes the advantage of the volume comparison of geodesic spheres, hence it assumes the manifold to have a pole. Very recently in [10] the authors seemed to be able to relate the conformal infinity and geodesic spheres and obtained a nice rigidity theorem for asymptotically hyperbolic with C^2 conformal compactifications. Hence it is easily seen that Theorem 1.2 becomes a significant step stone to utilize the regularity theorem in [8] of the conformally compact Einstein metrics to apply any available rigidity result. Let us first state and prove a rigidity theorem as an easy consequence of Theorem 1.2.

Theorem 5.1. *Suppose that $(X^{n+1},\ g)$ is a complete Einstein manifold with $Ric = -ng$. And suppose that it has an essential set $\mathbf{E}$ and that it satisfies the curvature condition (1) with $a > 2$. Also Suppose that X^{n+1} is simply connected at the infinity. Then $(X^{n+1},\ g)$ is a standard hyperbolic space if $4 \leq n \leq 6$ or X^{n+1} is spin if $n \geq 7$. And it is a standard hyperbolic space if in addition we assume that $\int_X \|Rm - \mathbf{K}\| d\mu_g < \infty$ if $n = 3$.*

Proof. Because of (1) and Einstein equations, applying Theorem 4.3 in [3], we see that

$$\|\nabla Rm\| \leq Ce^{-a\rho}.$$

Then by Theorem 1.2 proven in §3, we see that $\bar{g}$ is $C^{2,\mu}$ up to the infinity boundary for some $\mu \in (0,1)$.

On the other hand, due to the curvature condition (1) with $a > 2$ (plus $\int_M \|Rm - \mathbf{K}\| d\mu_g < \infty$ for $n = 4$) and Theorem 2.6 in [17], we see that the conformal infinity $(M^n, [\hat{g}])$ of $(X^{n+1},\ g)$ is locally conformally flat. Then by simply connectedness of X at the infinity we see that the boundary M^n of X is simply connected. Hence the conformal infinity $(M^n, [\hat{g}])$ is conformally equivalent to the standard sphere. As $\bar{g}$ is $C^{2,\mu}$ smooth at the infinity boundary, we know that there is a positive function $u \in C^{2,\mu}(M^n)$ so that $u^{\frac{4}{n-3}}\bar{g}$ is the metric of the standard sphere, i.e. u satisfies

$$\Delta_{\bar{g}} u - \frac{n-2}{4(n-1)} Ru + \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} = 0,$$

By setting

$$u(\rho, \theta) = u(\theta),$$

we may assume u is defined on X, hence $u^{\frac{4}{n-2}} \cdot \bar{g}$ is $C^{2,\mu}$ smooth near the infinity boundary M and its restriction on M is the standard sphere metric. Now due to Theorem A in [8], we know that in fact $(X^{n+1},\ g)$ is conformally compact of order C^∞. Then the theorem follows from the results in [15] and [1]. $\square$

To get rigidity results for asymptotically hyperbolic manifolds with only $Ric \geq -ng$ we need to have the following technical lemma, which is an improvement of Lemma 5.1 in [14], based on an idea from [8]. Namely,

Lemma 5.2. *Suppose $(X^{n+1},\ g)$ is a conformally compact manifold of regularity $C^{2,\lambda}$ for $\lambda \in (0,1)$. Then, for any $C^{2,\lambda}$ defining function r and $\bar{g} = r^2 g \in C^{2,\lambda}$, there is a $C^{2,\lambda}$ geodesic defining function x such that*

$$x^2 g = r^2 g = \bar{g}$$

on the boundary ∂X and $x^2 g \in C^{2,\lambda}$.

Proof. Let $\bar{g} = r^2 g$ and the coordinates near the boundary as $(\theta^0, \theta^1, \cdots, \theta^n)$, where $\theta^0 = r$. To find the geodesic defining function x we set $x = e^u r$ and consider the equations, as in [14],

$$F(\theta, du) = 2 < du, dr >_{\bar{g}} + r|du|^2_{\bar{g}} - \frac{1 - |dr|^2_{\bar{g}}}{r} = 0$$

in a neighborhood of the boundary. We are using the method of characteristics to solve this PDE, hence we turn to solve the system of ODE

$$\begin{cases} \dot{p} = -D_\theta F(\theta(t), p(t)) \\ \dot{z} = D_p F(\theta(t), p(t) \cdot p(t) \\ \dot{\theta} = D_p F(\theta(t), p(t)), \end{cases}$$

where $\theta(t)$ is the characteristic curve for the PDE, $u(\theta(t)) = z(t)$ and $du(\theta(t)) = p(t)$. Readers are refereed to the book [9] for the method of characteristics to solve first order nonlinear PDEs. We know the somehow trouble term is

$$\frac{1 - |dr|^2_{\bar{g}}}{r} \in C^{1,\lambda},$$

which was considered in the proof of Lemma 5.1 in [14] to be responsible for the loss of regularity.

First the so-called non-characteristic nature of the PDE is meant that

$$D_{p_0} F(0, \theta^1, \cdots, \theta^n, p_0, 0, \cdots, 0) = 2 \neq 0$$

at an admissible initial date set $(0, \theta^1, \cdot, \theta^n, 0, p_0, 0, \cdots, 0)$ for $(\theta(0), z(0), p(0))$. But here it is rather explicit that

$$p_0(\theta^1, \theta^2, \cdots, \theta^n) = \frac{1}{2}\frac{1 - |dr|^2_{\bar{g}}}{r}|_{r=0} \in C^{2,\lambda}.$$

We now consider u as a function of variables $(t, p^1, p^2, \cdots, p^n)$. We easily see that

$$\partial_t^2 u = \partial_t \dot{z} = (D_\theta D_p F \cdot \dot{\theta}) \cdot p + D_p F \cdot \dot{p} \in C^{0,\lambda}$$

and

$$\partial_t du = \dot{p} \in C^{0.\lambda}.$$

It is then left only to verify that

$$\partial_{\theta^\alpha} \partial_{\theta^\beta} u = \partial_{\theta^\alpha} \partial_{\theta^\beta} z = \partial_{\theta^\alpha} p_\beta \in C^{0,\lambda},$$

which equivalently is to verify that the solution of the system of ODE smoothly depends on the initial data up to certain order. Here, thanks to [8], we take a change of variable that

$$s = \log r.$$

Then the function $F \in C^{2,\lambda}$ with respect to the variables

$$(s, \theta^1, \cdots, \theta^n, p_0, p_1, \cdots, p_n).$$

(please see the proof of Lemma 6.1 in [8]) Therefore there is no loss of regularity with respect to the variables $(\theta^1, \theta^2, \cdots, \theta^n)$. Finally, note that $F \in C^{2,\lambda}$, then by Implicit Function Theorem we see that $u \in C^{2,\lambda}$ which implies x is $C^{2,\lambda}$ too. Thus the proof is finished. $\square$

We are now ready to state and show a rigidity theorem for asymptotically hyperbolic manifolds with $Ric \geq -ng$.

Theorem 5.3. *Suppose that* $(X^{n+1},\ g)$ *is a complete manifold with* $Ric \geq -ng$*. And suppose that it has an essential set* $\mathbf{E}$ *and that it satisfies the curvature condition (1) with* $a > 0$ *and (2) with* $k = 1$, $b > 2$*. Also Suppose that* X^{n+1} *is simply connected at the infinity. Then* $(X^{n+1},\ g)$ *is a standard hyperbolic space if* $4 \leq n \leq 6$ *or* X^{n+1} *is spin if* $n \geq 7$*. And it is a standard hyperbolic space if in addition we assume that* $\int_X \|Rm - \mathbf{K}\| d\mu_g < \infty$ *if* $n = 3$*.*

Proof. First we know from Theorem 1.2 in §3 that $\bar{g}$ is $C^{2,\mu}$ up to the infinity boundary for some $\mu \in (0, 1)$. And by the curvature condition (1) with $a > 2$ (plus $\int_X \|Rm - \mathbf{K}\| d\mu_g < \infty$ for $n = 3$) and Theorem 2.6 in [17], we see that the conformal infinity $(M^n, [\hat{g}])$ of $(X^{n+1},\ g)$ is locally conformally flat. Hence the conformal infinity $(M^n, [\hat{g}])$ is conformally equivalent to the standard sphere due to the simply connectedness at the infinity. As $\bar{g}$ is $C^{2,\mu}$ smooth at the infinity boundary, we know that there is a positive function $u \in C^{2,\mu}(M^n)$ so that $u^{\frac{4}{n-3}}\bar{g}$ is the metric of the standard sphere, i.e. u satisfies

$$\Delta_{\bar{g}} u - \frac{n-2}{4(n-1)} Ru + \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} = 0,$$

By setting

$$u(\rho, \theta) = u(\theta),$$

we may assume u is defined on X at least near the infinity, hence $u^{\frac{4}{n-2}} \cdot \bar{g}$ is $C^{2,\mu}$ smooth near the infinity boundary M and its restriction on M is the standard sphere metric.

Next, due to Lemma 5.2, there is a geodesic defining function $x \in C^{2,\mu}$ associated with the standard sphere metric of the conformal infinity. Therefore the proof of Theorem 1.1 in [6] works here. Note that why Theorem 1.1 in [6] requires the regularity to be $C^{3,\alpha}$ is due to the same assumptions in Lemma 5.1 in [14], which has been improved by the above Lemma 5.2. $\square$

Theorem 5.1 is a rigidity theorem for Einstein AH manifolds; while Theorem 5.3 requires the curvature condition (2) though it no longer needs the Einstein equations. And in both cases the rigidity in higher dimensions requires the spin condition. We noticed the recent work of Dutta and Javaheri [10] where no spin condition is assumed. The argument in [10] is based on the volume comparison argument in [17] for AH manifolds with conformal compactification of C^2 regularity and an additional assumption that

$$R + n(n+1) = o(e^{-2\rho}) \tag{31}$$

where R is the scalar curvature. We will use our curvature estimates and regularity theorems to replace the C^2 regularity assumption in [10] to prove Theorem 1.6. Since the proof follows the approach in [10] with a number of modifications, we will sketch a proof in the following for readers′ conveniences. Hence from now on we will work with asymptotically hyperbolic manifold $(X^{n+1},\ g)$ that satisfy all assumptions in Theorem 1.6.

As in [10], let p_0 be any point in X, $t(x)$ be the distance function to p_0 with respect to metric g, $C(p_0)$ be the cut locus of p_0 in (X,g), Σ_t be the geodesic sphere of p_0 with radius t in (X,g), $\bar{g}=\sinh^{-2}\rho\cdot g$, $h=\sinh^{-2}t\cdot g$. Let γ_t, η_t be the restriction metric of $\bar{g}$ and h on Σ_t respectively. We continue to use ρ as before to stand for the distance to the essential set $\mathbf{E}$ and let $u=t-\rho$. It is clear that u is bounded.

Because of (8) in §2 we have Lemma 2.1 in [10] valid even without C^2 regularity of the conformal compactness. Hence we immediately have

Lemma 5.4. *There is constant Λ which is independent of t such that*

$$\|\nabla_g u\|\leq \Lambda e^{-\rho},$$

which is equivalent to

$$\|\nabla_{\bar{g}} u\|\leq \Lambda,$$

whenever t is smooth.

Proof. Let $\phi(t)=g(\nabla\rho,\nabla t)$. Then

$$g(\nabla u,\nabla u)=2(1-\phi).$$

To estimate ϕ, as in [10], we notice that

$$\partial_t\phi=(1-\phi^2)\nabla^2\rho(n,n),$$

where one writes $\nabla t=\phi\nabla\rho+\sqrt{1-\phi^2}n$ and n is a unit vector that is perpendicular to $\nabla\rho$. Hence in the light of (8) one gets

$$\partial_t\phi=(1-\phi^2)(1+O(e^{-2t})).$$

By the proof of Lemma 2.1, we then get

$$\phi=1+O(e^{-2t}). \tag{32}$$

and finish the proof. □

An important consequence of the above lemma, as observed in [10], is that the limit of the function u is a Lipschitz function on the infinity as $t\to\infty$, in $W^{1,p}$-norm for any $p>1$. A geodesic in $(X,\ g)$ is said to be a ρ-geodesic if it is a geodesic emanated from $\mathbf{E}$ and is perpendicular to $\partial\mathbf{E}$; a geodesic is said to be a t-geodesic if it is geodesic which is a geodesic ray from p_0. The following lemma is Corollary 3.2 in [10] which is another straightforward consequence of the above Lemma 5.4.

Corollary 5.5. *There is $\rho_0 > 0$ such that in the region $\rho > \rho_0$ such that the function $t(x)$ is increasing along the ρ-geodesics and the function $\rho(x)$ is increasing along the t-geodesics.*

Proof. Suppose that x_1 and x_2 are two points in a ρ-geodesic with distance s, i.e.

$$\rho(x_1) - \rho(x_2) = s > 0.$$

Then

$$\begin{aligned} t(x_1) - t(x_2) &= s + u(x_1) - u(x_2) \\ &= s + sg(\nabla u, \nabla \rho) \\ &\geq s(1 - \Lambda e^{-2\rho}). \end{aligned} \tag{33}$$

Hence there is ρ_0 such that $t(x)$ is increasing along ρ-geodesics where $\rho > \rho_0$. Similarly we may show that ρ is increasing along t-geodesics where $\rho > \rho_0$ (set ρ_0 bigger if necessary). Thus the proof of the lemma is finished. □

Analogue to our previous rigidity theorems in this section we know the asymptotically hyperbolic manifolds that satisfy all the assumptions in Theorem 1.6 are conformally compact of regularity $C^{1,\alpha}$(or $W^{2,p}$) due to our Theorem 1.5 and have the standard round sphere as the conformal infinities. Particularly we know that $\partial \mathbf{E}$ is diffeomorphic to $\mathbf{S}^n$. One of the main observation in [10] is the following lemma, whose proof still holds with no modification.

Lemma 5.6. *(Lemma 4.1 in* [10]*) For t large enough, Φ_t: $\mathbf{S}^n \mapsto \Sigma_t$ is a homeomorphism. Moreover it is a local diffeomorphism at $\theta \in \mathbf{S}^n$ where $\Phi_t(\theta) \notin C(p_0)$.*

In fact the set $\{\theta \in \mathbf{S}^n : \Phi_t(\theta) \notin C(p_0)\}$ is rather negligible when we are concerned with the integrals. Let μ_0 be the standard metric on $\mathbf{S}^n$.

Lemma 5.7. *For almost all t, when large enough, $\Phi_t^{-1}(\Sigma_t \cap C(p))$ is measure zero in $(\mathbf{S}^n, \mu_0)$.*

Proof. At least when ρ is large enough, we may consider the map

$$\Lambda(x) = (\Pi(x), t(x)) : X \setminus E \mapsto \mathbf{S}^n \times [0, \infty),$$

given by the exponential map from $\partial \mathbf{E}$ by the nature of an essential set and the monotonicity of the function t along each ρ-geodesics. Note that Π and t are Lip, so is Λ. Therefore $\Lambda(C(p_0))$ is measure zero in $\mathbf{S}^n \times [0, \infty)$. Due to Fubini Theorem, we see that for almost all t, when large enough, $\mathbf{S}^n \times \{t\} \cap \Lambda(C(p_0))$ is zero measure. In the light of the fact

$$\Lambda|_{\Sigma_t} = \Phi_t^{-1}|_{\Sigma_t},$$

the lemma is then proven. □

As argued in [17] and [10], due to Gromov-Bishop volume comparison theorem, to prove Theorem 1.6 it suffice to show

$$\lim_{t\to\infty} Vol(\Sigma_t, \eta_t) \geq \omega_n,$$

where ω_n is the volume of the standard sphere $\mathbf{S}^n$. To this purpose, we study the pull back metric $(\Phi_t^{-1})_*\eta_t$ on $\mathbf{S}^n \setminus \Phi_t^{-1}(C(p_0))$ as t approaches to the infinity. Note that

$$(\Phi_t^{-1})_*\eta_t = 4e^{-2u}(\Phi_t^{-1})_*(\bar{g}|_{\Sigma_t}),$$

and Σ_t can be expressed as a graph $(\theta, f(\theta))$ on $\mathbf{S}^n$. Hence we have

$$\frac{\partial}{\partial t} = (1 + |\nabla_g f|^2)^{-\frac{1}{2}}(\frac{\partial}{\partial \rho} - g^{ij}\frac{\partial f}{\partial \theta^i}\frac{\partial}{\partial \theta^j}),$$

which, together with (32), implies

$$|\nabla_g f|^2 = O(e^{-2\rho}).$$

Therefore we see that

$$(\Phi_t^{-1})_*(\bar{g}|_{\Sigma_t}) = \bar{g}_{ij}(t,\theta)d\theta^i d\theta^j + O(e^{-2\rho}).$$

Thus

$$\lim_{t\to\infty}(\Phi_t^{-1})_*(\eta_t) = \lim_{\rho\to\infty} 4e^{-2u}\bar{g}|_{\Sigma_\rho} \triangleq \eta_0,$$

where $\eta_0 = v^{\frac{4}{n-2}}\mu_0$ and v is Lipschtz on $\mathbf{S}^n$ satisfying

$$n(n-1)\omega_n^{\frac{2}{n}} \leq \frac{\int_{\mathbf{S}^n}(\frac{4(n-1)}{(n-2)}|\nabla_{\mathbf{S}^n} v|^2 + n(n-1)v^2)d\mu_0}{(\int_{\mathbf{S}^n} v^{\frac{2n}{n-2}}d\mu_0)^{\frac{n-2}{n}}},$$

since the minimum of the Yamabe functional on $\mathbf{S}^n$ is $n(n-1)\omega_n^{\frac{2}{n}}$.

Now, on one hand, if denote $\eta_\rho = w^{\frac{4}{n-2}}\bar{g}|_{\Sigma_\rho}$, $w = e^{\frac{2-n}{2}u}$, and $\bar{g}_\rho = \bar{g}|_{\Sigma_\rho}$, we have

$$\begin{aligned}\lim_{\rho\to\infty}\frac{\int_{\mathbf{S}^n} R_{\eta_\rho}d\eta_\rho}{(\int_{\mathbf{S}^n} d\eta_\rho)^{\frac{n-2}{n}}} &= \lim_{\rho\to\infty}\frac{\int_{\mathbf{S}^n}(\frac{4(n-1)}{(n-2)}|\nabla_{\bar{g}_\rho} w|^2 + R_{\bar{g}_\rho}w^2)d\bar{g}_\rho}{(\int_{\mathbf{S}^n} w^{\frac{2n}{n-2}}d\bar{g}_\rho)^{\frac{n-2}{n}}} \\ &= \frac{\int_{\mathbf{S}^n}(\frac{4(n-1)}{(n-2)}|\nabla_{\bar{g}_0} w|^2 + R_{\bar{g}_0}w^2)d\bar{g}_0}{(\int_{\mathbf{S}^n} w^{\frac{2n}{n-2}}d\bar{g}_0)^{\frac{n-2}{n}}} \\ &= \frac{\int_{\mathbf{S}^n}(\frac{4(n-1)}{(n-2)}|\nabla_{\mathbf{S}^n} v|^2 + n(n-1)v^2)d\mu_0}{(\int_{\mathbf{S}^n} v^{\frac{2n}{n-2}}d\mu_0)^{\frac{n-2}{n}}} \\ &\geq n(n-1)\omega_n^{\frac{2}{n}}.\end{aligned} \tag{34}$$

Because $\bar{g}$ is $W^{2,p}$-regular up to the boundary of $(X, \bar{g})$ due to Theorem 1.5 and the comment right after the proof of Lemma 5.4. On the other hand, by

direct computations (please see the calculations in p.556 in [17]), we recall that,

$$R_{\eta_\rho} \leq n(n-1) + o(1).$$

Therefore we obtain

$$Vol(\mathbf{S}^n, \eta_0) = \lim_{\rho \to \infty} Vol(\mathbf{S}^n, \eta_\rho) \geq \omega_n,$$

which implies

$$\lim_{t \to \infty} Vol(\Sigma_t, \eta_t) \geq \omega_n.$$

Thus the proof of Theorem 1.6 is complete.

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